Ergodic Theory - Week 11

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1 Spectral theory of measure-preserving systems

P1. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and let $f \in L^2(X)$ be an eigenfunction with eigenvalue $e(\alpha)$ for some $\alpha \in [0,1)$. Calculate the spectral measure μ_f of f.

P2. Find an eigenfunction and a weak-mixing function for the system $(\mathbb{T}^2, \mathcal{B}(\mathbb{T}^2), m_{\mathbb{T}^2}, T)$, where $T(x,y) = (x+\alpha, x+y)$, for some $\alpha \in [0,1)$. Do the same for the system on the same space but with the map $S(x,y) = (x+\alpha, 2y)$.

P3. Given a measure preserving system $(X, \mathcal{B}, \mu, T)$, show that for any $f, g \in L^2(X)$ with f weak-mixing, $f \otimes g \in L^2(X \times X)$ is weak-mixing.

P4. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and consider the space

$$\mathcal{H}_c = \overline{\operatorname{span}\{f \in L^2(X) \colon f \text{ is an eigenfunction}\}}.$$

Show that if $f \in \mathcal{H}_c$, then the closure of the orbit $\{U_T^n f \colon n \in \mathbb{N}\}$ in the $L^2(X)$-norm is compact (for this reason, functions in $\mathcal{H}_c$ are called compact functions or almost-periodic functions).

Hint: By completeness, it suffices to show that the orbit $\{U_T^n f \colon n \in \mathbb{N}\}$ is totally-bounded: for any $\varepsilon > 0$, there exists a finite collection of functions in $g_1, \ldots, g_m \in L^2(X)$, such that for every $n \in \mathbb{N}$, we have

$$\min_{1 \le i \le m} \left\| U_T^n f - g_i \right\|_{L^2(X)} < \varepsilon.$$